MA 681, Spring 2015

Assignment 8.

Cauchy's Integral Formula (mostly).

This assignment is due Wednesday, March 25 (because of spring break). Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

In all problems below, L denotes a closed rectifiable simple curve, traversed counterclockwise, with interior $I(L)$ and exterior $E(L)$.

(1) Evaluate the integral using Cauchy's Integral Formula:
$$\int_{|z-a|=a} \frac{z}{z^4-1}\,dz,$$
where $a \in \mathbb{R}, a > 1$. (*Hint:* Represent the integrand as $\frac{f(z)}{z-z_0}$.)

(2) Evaluate the integral
$$\frac{1}{2\pi i}\int_L \frac{ze^z}{(z-z_0)^3}\,dz,$$
where $z_0 \in I(L)$. (*Hint:* Use formula for derivative of Cauchy's integral.)

(3) *Cauchy's formula for an unbounded domain.* Let L be a closed rectifiable simple curve, traversed counterclockwise. Let $f(z)$ be a differentiable function on a domain G, where $L \cup E(L)$ is contained in G. (That is, f is differentiable on a neighborhood of L and *outside* of L, but not necessarily inside.) In particular, $f(z)$ is differentiable at infinity, that is $f(1/w)$ is differentiable at $w = 0$. Suppose that
$$\lim_{z\to\infty} f(z) = A.$$
Prove that
$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}\,d\zeta = A, \quad \text{if } z \in I(L),$$
and
$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}\,d\zeta = -f(z) + A, \quad \text{if } z \in E(L).$$
(*Hint:* Pass to intergation over a large circle $|z| = R$ using Cauchy theorem for multiple contours. Then either use Problem 5 of HW6 (shorter way), or the substitution $z = 1/w$, $\zeta = 1/\eta$ (longer way).)

(4) Let $f(z) = \frac{2013z^{12}-z^3+3102z^2+100}{(z-1)^3(z-2)^4(z-3)^2(z-4)(z-5)^2}$. Evaluate
$$\int_L \frac{f(\zeta)}{\zeta - z_0}\,d\zeta,$$
if $I(L)$ contains the disc $|z| \le 5$, and $z_0 \ne 1, 2, 3, 4, 5$. (*Hint:* Use the problem above.)

NOTE that the "finite" Cauchy's formula is also usable, but much messier in this case.

(5) Find a harmonic conjugate for the following functions. (If one does not exist, show that.)
  (a) $4x^3y - Axy^3$, where A is a real number.
  (b) $\sin y \sinh x$.
  (c) $e^{-2xy}\sin(x^2 - y^2)$ (*Hint:* Use the statement about composition of harmonic functions.)